Gaussian Elimination for those who have completed Linear Algebra

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I. INTRODUCTION

et was immensely frustrating to have to retake linear algebra multiple times in my life. It was even more frustrating to have forgotten computation rules for gaussian elimination, and how one uses them to find properties of the linear transformation such as rank, null space or solution space. The issue is as follows:

Gaussian elimination is almost always taught as the first subject of any linear algebra course, followed by the interpretation of a matrix as a linear transformation. After such an interpretation is given and the isomorphism defined, courses rarely return to the gaussian elimination algorithm, to explain what happens to the linear transformation when gaussian elimination is performed. I hope to bridge this gap in this text.

In this text, T shall always be a linear transformation $T : V \to W$, where V and W are vector spaces over a field $\mathbb{F}$. We will denote a basis for both as $\{v_1, v_2, \dots, v_n\}$ and $\{w_1, w_2, \dots, w_m\}$

II. ROW/COLUMN OPERATIONS

The row/column operations correspond to changing the basis of either the codomain or domain.

Column Operations Column operations affect the basis of the *domain*.

1. Swapping two columns i and j swaps the basis vectors v_i and v_j.

2. Multiplying column i by λ corresponds to multiplying the vector v_i by λ.

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3. Adding k times of column i to column j corresponds to adding $v_j \to v_j + kv_i$

Row Operations Row operations affect the basis of the codomain.

1. Swapping two rows i and j swaps w_i and w_j.

2. Multiplying row i by λ corresponds to multiplying w_i by λ^{-1}.

3. Adding k times of row i to row j corresponds to setting $w_i \to w_i - kw_j$ (notice that i and j flip roles here!)

It is clear that the fundamental linear transformation is not affected here, because all the row/column operations can be expresed as a change of basis. In particular, it preserves the rank and kernel of the linear transformation.

> **IMPORTANT FACT:** Row/column operations on a matrix can be interpreted as follows: The underlying linear transformation is unchanged, while the bases of either the domain or codomain are changed.

III. REDUCED ROW ECHELON FORM

What the Reduced Row Echelon Form is is well described in many linear algebra books. I will simply work with an example of a matrix and its reduced row echelon form:

$$[T] = M := \begin{pmatrix} 1 & 2 & 1 & 5 \\ 2 & 1 & -1 & 4 \\ 1 & 0 & -1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \stackrel{\text{def}}{=} M',$$

where T is the linear transformation, and M and M' are the matrices in the two different bases. Recall that since we get from the original matrix to its reduced row echelon form by row operations, this corresponds simply to a change of basis of the codomain, not any change of basis of the domain. Let the basis in which the linear transformation takes the form M' be denoted by $\{w'_1, w'_2, \ldots, w'_m\}$ (in this particular example $m = 4$)

I shall also not explain in this text what an augmented matrix is.

IV. SOLUTION SPACE

It is clear by simply writing out the equations why the solution space for $Ax = b$ is the same as the solution space for $\tilde{A}x = \tilde{b}$, if one can get from $[A|b]$ to $[\tilde{A}|\tilde{b}]$ by elementary row operations.

A. No Solutions

Now suppose we have reduced $[A|b]$ to reduced row echelon form, and get

$$\left(\begin{array}{cccc|c} 1 & \dots & \dots & \dots & \dots \\ 0 & 1 & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & 1 \end{array}\right).$$

This equation clearly has no solutions, since the bottom equation is $0 = 1$.

B. Unique Solution

Clearly, if $\tilde{A} = I$, $\tilde{b}$ is the unique solution.

C. Infinite Solutions

If there are nonpivotal unknowns, then for each value of the nonpivotal unknowns, there is a solution to the system of linear equations. This is clear from choosing a value for each of the nonpivotal unknowns, and "moving it to the other side of the equation".

V. IMAGE & KERNEL

In this section, we will work with the example from earlier.

$$[T] = M := \begin{pmatrix} 1 & 2 & 1 & 5 \\ 2 & 1 & -1 & 4 \\ 1 & 0 & -1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \stackrel{\text{def}}{=} M'.$$

A. Image Space

It is clear that the images of the vectors v_1 and v_2 span the entire image space, because their images are w_1' and w_2', and we read off clearly that all other image vectors $T(v_3)$ and $T(v_4)$ can be expressed in terms of $T(v_1)$ and $T(v_2)$.

However, when the "basis of the image space" is requested in linear algebra exams, this usually means that we are to find a basis for the image space and express it in terms of the original basis that we were given. Thus we cannot use any columns from the reduced matrix. Instead, we use the fact that $T(v_1)$ and $T(v_2)$ form a basis, and thus read off the image space

$$\operatorname{im}(T) = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

B. Kernel

The kernel is simply the solution of the system of linear equations $T(v) = 0$. Recalling that the solution space for any matrix representation of T related by row operations is identical. Then we simply have to solve $M'v = 0$. This is made significantly easier by the matrix being in reduced row echelon form; it is however not not required. Recall that we can let each nonpivotal unknown be a free parameter. Thus:

$$\begin{aligned} x_1 =& x_3 - x_4 \\ x_2 =& - x_3 - 2x_4 \end{aligned}$$

It is now simple enough to come up with vectors in the kernel. But for a basis, we need to choose 2 vectors and ensure that they are linearly independent. This is easiest done by choosing a vector with $(x_3, x_4) = (1, 0)$, and one more with $(x_3, x_4) = (0, 1)$. This gives a basis for the kernel

$$\ker(T) = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -2 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

VI. CONCLUSION & ACKNOWLEDGEMENTS

I would like to thank math stackexchange for informing me about the meaning of row operations, in particular this post (although I think they mixed up row and column operations).

I would also like to thank the book by Hubbard & Hubbard (Vector Calculus, Linear Algebra & Differential Forms). I think that their treatment of linear algebra, with the definition of a matrix as a linear transformation coming first, should be seen as an ideal to be aspired towards.